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# LETTERS TO THE EDITOR

## NON-CLASSICAL BOUNDARY CONDITIONS AND DQM

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### 1. Introduction

The differential quadrature method (henceforth DQM) seems to be a promising numerical tool for analyzing differential equations with boundary and/or initial conditions.

At the early stage of the method, the satisfaction of the Neumann boundary conditions in fourth order systems was rather intriguing, and the so-called $\delta$ approach, as proposed by Bert and coworkers (see, e.g., reference [1]), turned out to be approximate and not completely reliable. Moreover, sometimes it produced badly conditioned matrices, with consequent numerical inaccuracies. More recently, an improved method [2] allowed one to satisfy exactly all the boundary conditions in a fourth order system, and a straightforward generalization of this approach [3] somewhat simplified the analysis in the presence of classical boundary conditions of the Dirichlet and Neumann type. Another, powerful generalization should also be mentioned [4, 5].

In this letter, the DQM is applied to dynamic and stability analysis of beams with non-classical boundary conditions, the obtained results are compared with the exact frequencies and critical loads, and the agreement is shown to be quite satisfactory for the entire parameter range.

### 2. The structural system

Consider a beam with span $L$, Young modulus $E$, second moment of area $I$, mass density $\rho$ and cross-sectional area $A$. The beam ends are both elastically constrained against the vertical displacements and rotations, with vertical flexibilities at left and right $c_{vl}$, $c_{vr}$, respectively, and rotational flexibilities $c_{rl}$, $c_{rr}$.

The equation of motion of the beam in the presence of an axial force $F$ at the right end can be written as

$$EI\,\partial^4 v/\partial x^4 + (F - k_p)\,\partial^2 v/\partial x^2 + k_w v - \rho A\omega^2 v = 0, \tag{1}$$

where $v(x)$ is the transverse displacement, the parameters $k_w$ and $k_p$ define a two-parameter elastic soil, and $\omega^2$ denotes the free vibration frequency of the beam.

The boundary conditions are

$$EI\,\partial^3 v/\partial x^3|_{x=0} = -(1/c_{vl})v(0), \qquad EI\,\partial^2 v/\partial x^2|_{x=0} = (1/c_{rl})\,\partial v/\partial x|_{x=0},$$

$$EI\,\partial^3 v/\partial x^3|_{x=L} + (F - k_p)\,\partial v/\partial x|_{x=L} = (1/c_{vz})v(L),$$

$$EI\,\partial^2 v/\partial x^2|_{x=L} = -(1/c_{rl})\,\partial v/\partial x|_{x=L}. \tag{2}$$

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It is convenient to map the physical domain $[0, L]$ on to the natural Gaussian domain $[-1, 1]$, by means of the transformation

$$\xi(x) = 2(x/L) - 1, \tag{3}$$

where $x$ is the Cartesian co-ordinate and $\xi$ its natural counterpart.

It follows that the differential equation becomes

$$\partial^4 v(\xi)/\partial\xi^4 + (\lambda - \kappa_p)\, \partial^2 v(\xi)/\partial\xi^2 + \kappa_w v(\xi) - \Omega^2 v(\xi) = 0, \tag{4}$$

where

$$\lambda = FL^2/4EI, \qquad \kappa_p = k_p L^2/4EI, \qquad \kappa_w = k_w L^4/16EI, \qquad \Omega^2 = \rho A\omega^2 L^4/16EI. \tag{5}$$

The non-dimensional boundary conditions are given by

$$\partial^3 v/\partial\xi^3|_{\xi=-1} = -(1/\chi_{vl})v(-1), \qquad \partial^2 v/\partial\xi^2|_{\xi=-1} = (1/\chi_{rl})\,\partial v/\partial\xi|_{\xi=-1},$$
$$\partial^3 v/\partial\xi^3|_{\xi=1} + (\lambda - \kappa_p)\,\partial v/\partial\xi|_{\xi=1} = (1/\chi_{vz})v(1), \qquad \partial^2 v/\partial\xi^2|_{\xi=1} = -(1/\chi_{rl})\,\partial v/\partial\xi|_{\xi=1}, \tag{6}$$

where the non-dimensional axial flexibilities and rotational flexibilities can be expressed as

$$\chi_{vl} = 8EIc_{vl}/L^3, \qquad \chi_{rl} = 2EIc_{rl}/L, \qquad \chi_{vz} = 8EIc_{vz}/L^3, \qquad \chi_{rr} = 2EIc_{rr}/L. \tag{7}$$

## 3. A BRIEF OVERVIEW OF THE METHOD

In order to discretize the differential equation of motion, the natural interval is divided into $n$ segments defined by means of $n+1$ points located at the abscissae $\xi_1, \xi_2, \ldots, \xi_{n+1}$. One can assume the set of $(n+7)$ nodal unknowns

$$\mathbf{d}^{\mathrm{T}} = \{u_1,\ u_1',\ u_1'',\ u_1''',\ u_2,\ \cdots,\ u_{n+1},\ u_{n+1}',\ u_{n+1}'',\ u_{n+1}'''\}, \tag{8}$$

and the displacement $v(\xi)$ of the beam can be approximated as

$$v(\xi) = \boldsymbol{\alpha}\mathbf{C} = \sum_{i=1}^{n+7} \alpha_i C_i, \tag{9}$$

where $\boldsymbol{\alpha}$ is a row vector of monomials, and $\mathbf{C}$ is a column vector of Lagrangian co-ordinates. From equation (9) it is easily seen that

$$v'(\xi) = \boldsymbol{\alpha}'\mathbf{C}, \qquad v''(\xi) = \boldsymbol{\alpha}''\mathbf{C}, \qquad v'''(\xi) = \boldsymbol{\alpha}'''\mathbf{C}, \tag{10}$$

and therefore

$$\mathbf{d} = \left\{ \begin{array}{c} \alpha_1 \\ \alpha_1' \\ \alpha_1'' \\ \alpha_1''' \\ \alpha_2 \\ \vdots \\ \alpha_{n+1}''' \end{array} \right\} = \mathbf{N}_0 \mathbf{C}. \tag{11}$$

Following the same approach as in reference [1], one can define the weighting coefficients of the first four derivatives, as follows:

$$\mathbf{A} = \mathbf{N}_0'\mathbf{N}_0^{-1}, \qquad \mathbf{B} = \mathbf{AA}, \qquad \mathbf{C} = \mathbf{AAA}, \qquad \mathbf{D} = \mathbf{AAAA}. \tag{12}$$

The discretized version of equation (4) is

$$\begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,n+7} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,n+7} \\ L_{3,1} & L_{3,2} & \cdots & L_{3,n+7} \\ L_{4,1} & L_{4,2} & \cdots & L_{4,n+7} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+7,1} & L_{n+7,2} & \cdots & L_{n+7,n+7} \end{bmatrix} \begin{bmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{bmatrix} = \Omega^2 \begin{bmatrix} v_1 \\ v_1' \\ v_1'' \\ v_1''' \\ \vdots \\ v_{n+1}''' \end{bmatrix}, \tag{13}$$

where the matrix $\mathbf{L}$ is the discretized version of the differential operator

$$\mathscr{L} = \partial^4/\partial\xi^4 + (\lambda - \kappa_p)\,\partial^2/\partial\xi^2 + \kappa_w, \tag{14}$$

and, as such is given by

$$L_{ij} = D_{ij} + (\lambda - \kappa_p)B_{ij} + \kappa_w\delta_{ij}, \tag{15}$$

where $\delta_{ij}$ is the well-known Kronecker operator.

In order to impose the boundary conditions, it is now convenient to interchange the rows (and columns) $(n+4)$ and $(n+5)$ of the matrix $\mathbf{L}$ with the third and fourth rows (and columns), so that the boundary conditions can be immediately imposed:

$$\left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -\chi_{vl} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & \chi_{rl} & 0 & 0 & 0 \\ 0 & 0 & 1 & -\chi_{vz}(\lambda - \kappa_p) & 0 & \cdots & 0 & 0 & 0 & -\chi_{vz} \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \chi_{rr} & 0 \\ \hline L_{5,1} & L_{5,2} & L_{5,n+4} & L_{5,n+5} & L_{5,5} & \cdots & L_{5,3} & L_{5,4} & L_{5,n+6} & L_{5,n+7} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{3,1} & L_{3,2} & L_{3,n+4} & L_{3,n+5} & L_{3,5} & \cdots & L_{3,3} & L_{3,4} & L_{3,n+6} & L_{3,n+7} \\ L_{4,1} & L_{4,2} & L_{4,n+4} & L_{4,n+5} & L_{4,5} & \cdots & L_{4,3} & L_{4,4} & L_{4,n+6} & L_{4,n+7} \\ L_{n+6,1} & L_{n+6,2} & L_{n+6,n+4} & L_{n+6,n+5} & L_{n+6,5} & \cdots & L_{n+6,3} & L_{n+6,4} & L_{n+6,n+6} & L_{n+6,n+7} \\ L_{n+7,1} & L_{n+7,2} & L_{n+7,n+4} & L_{n+7,n+5} & L_{n+7,5} & \cdots & L_{n+7,3} & L_{n+7,4} & L_{n+7,n+6} & L_{n+7,n+7} \end{array}\right]$$

$$\times \left[\begin{array}{c} v_1 \\ v_1' \\ v_{n+1} \\ v_{n+1}' \\ \hline v_2 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{array}\right] = \Omega^2 \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline v_2 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{array}\right]. \tag{16}$$

In partitioned form, the previous equations can be written as

$$\begin{pmatrix} \mathbf{L}_{aa} & \mathbf{L}_{ab} \\ \mathbf{L}_{ba} & \mathbf{L}_{bb} \end{pmatrix} \begin{pmatrix} \mathbf{w}_c \\ \mathbf{w} \end{pmatrix} = \Omega^2 \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}, \tag{17}$$

where $\mathbf{w}_c$ is the vector of the passive coordinates,

$$\mathbf{w}_c = \begin{Bmatrix} v_1 \\ v_1' \\ v_{n+1} \\ v_{n+1}' \end{Bmatrix}, \tag{18}$$

and $\mathbf{w}$ is the vector of the active co-ordinates,

$$\mathbf{w} = \begin{Bmatrix} v_2 \\ v_3 \\ \vdots \\ v_1'' \\ v_1''' \\ v_{n+1}'' \\ v_{n+1}''' \end{Bmatrix}. \tag{19}$$

The passive degrees of freedom can be easily condensed, and the following reduced eigenvalue problem is obtained:

$$(\mathbf{L}_{bb} - \mathbf{L}_{ba}\mathbf{L}_{aa}^{-1}\mathbf{L}_{ab})\mathbf{w} = \Omega^2\mathbf{w}. \tag{20}$$

It is perhaps worth noting that, in the absence of axial forces and elastic soil, no matrix inversion is involved in the condensation process, because in this case the matrix $\mathbf{L}_{aa}$ is given by an identity matrix.

## 4. NUMERICAL EXAMPLES

All computations for the numerical examples have been performed by using two different choices of the monomials $\alpha_i$. In the first case $\alpha_i = \xi^{i-1}$ and the sampling points are uniformly distributed along the natural interval

$$\xi_i = [2(i-1) - n]/n, \qquad i = 1, 2, \ldots, n+1. \tag{21}$$

In the second case $\alpha_i = T_{i-1}(\xi)$, where $T_i(\xi)$ are the Chebyshev polynomials of the first kind, and the sampling points are located at the so-called Gauss–Lobatto–Chebyshev points,

$$\xi_i = -\cos(\pi(i-1)/n), \qquad i = 1, 2, \ldots, n+1. \tag{22}$$

In Table 1 the first three nondimensional natural frequencies of vibration are reported, in the absence of axial loads and elastic soils, for $\chi_{vz} = 0$, $\chi_{rl} = 0$, $\chi_{rr} = 0$, and for various values of the non-dimensional vertical flexibility at left $\chi_{vl}$. The results have been obtained for $n = 8$, and are compared with the exact results, which in this particular case could be obtained by solving the frequency equation [6]. It is worth noting that the use of the Chebyshev polynomials implies a greater precision, especially for the higher frequencies. In any case, the agreement is quite satisfactory.

TABLE 1

*First three non-dimensional frequencies of a beam with flexible ends*

| $\chi_{vl}$ | | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ |
|---|---|---|---|---|
| 0 | Uniform grid | 22·37329 | 61·67194 | 120·85757 |
| | Chebyshev grid | 22·37329 | 61·67436 | 120·82882 |
| | Exact | 22·37329 | 61·67282 | 120·90339 |
| 0·001 | Uniform grid | 21·38780 | 53·11998 | 91·57713 |
| | Chebyshev grid | 21·38780 | 53·12383 | 91·52081 |
| | Exact | 21·38780 | 53·12356 | 91·51347 |
| 0·005 | Uniform grid | 17·75903 | 37·73705 | 77·64691 |
| | Chebyshev grid | 17·75903 | 37·73758 | 77·60824 |
| | Exact | 17·75901 | 37·73758 | 77·60118 |
| 0·01 | Uniform grid | 14·80957 | 33·89799 | 76·09249 |
| | Chebyshev grid | 14·80957 | 33·89823 | 76·05848 |
| | Exact | 14·80957 | 33·89823 | 76·05188 |
| 0·05 | Uniform grid | 8·884959 | 30·90089 | 74·94655 |
| | Chebyshev grid | 8·884959 | 30·90100 | 74·91615 |
| | Exact | 8·884959 | 30·90100 | 74·90992 |
| 0·1 | Uniform grid | 7·470370 | 30·55817 | 74·80982 |
| | Chebyshev grid | 7·470370 | 30·55827 | 74·77985 |
| | Exact | 7·470370 | 30·55827 | 74·77366 |
| 1 | Uniform grid | 5·813812 | 30·25851 | 74·68802 |
| | Chebyshev grid | 5·813812 | 30·25860 | 74·65844 |
| | Exact | 5·813812 | 30·25860 | 74·65230 |
| 10 | Uniform grid | 5·615816 | 30·22903 | 74·67590 |
| | Chebyshev grid | 5·615817 | 30·22912 | 74·64637 |
| | Exact | 5·615815 | 30·22912 | 74·64022 |
| 100 | Uniform grid | 5·595576 | 30·22608 | 74·67469 |
| | Chebyshev grid | 5·595575 | 30·22617 | 74·64516 |
| | Exact | 5·595575 | 30·22617 | 74·63902 |

In Table 2 the influence of the axial force on the free vibration frequency is illustrated for a beam clamped at the left and simply supported at the right with a flexible support ($\chi_{vl} = 0{\cdot}5$). Even in this case $n = 8$, whereas the support at the right is simulated by giving the large value $\chi_{rr} = 100\,000$ as the right rotational flexibility.

TABLE 2

*Free vibration frequencies versus axial load for a propped cantilever beam with a flexible support*

| $\lambda$ | $\Omega_1$, uniform grid | $\Omega_1$, exact |
|---|---|---|
| 0 | 4·49483 | 4·49482 |
| 1 | 3·96342 | 3·96342 |
| 2 | 3·30980 | 3·30980 |
| 3 | 2·42482 | 2·42482 |
| 4 | 0·63363 | 0·63364 |
| 4·05 | 0·33936 | 0·33937 |
| 4·07 | 0·01667 | 0·01680 |
| 4·070047 | 0·00279 | 0·00347 |

The performances of the differential quadrature method are not influenced by the value of the axial loads, which can reach its critical value without causing any numerical error.

## 5. CONCLUSIONS

The differential quadrature method has been applied to a class of one-dimensional boundary problems in the presence of non-classical boundary conditions. It is shown that the proposed approach satisfies exactly all the four boundary conditions, leading to a simple eigenvalue problem.

A small *Mathematica* notebook [7] was written, in order to compare the appropriate eigenvalues with the exact results, and some effort was made in order to minimize the round-off errors. Finally, numerical examples and comparisons show the effectiveness of the proposed method.

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